MATH2040 Linear Algebra II

Tutorial 11

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1 Examples:

Recall(Dot diagram): For each eigenvalue λ_i, let β_i be the basis of K_{λ_i}, then $\beta_i = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{n_i}$ is a disjoint union of cycles of generalized eigenvectors with $p_1 \geq p_2 \geq \cdots \geq p_{n_i}$, where p_j be the length of γ_j. Then, the dot diagram corresponding to λ_i has the following properties:

1. Number of columns in the dot diagram determines number of cycles in β_i
2. Number of dots in the j-th column determines number of element in γ_j
3. Since the cycles are arranged in descending order of length, so the number of dots in each column is non-increasing from left to right
4. Number of dots in the first r rows equals nullity$((T - \lambda_i I)^r)$
5. The total number of dots equal to $\dim(K_{\lambda_i}) = m_i$, which is the multiplicity of λ_i

Using these properties, we can find the dot diagrams systematically and can determine the structure of the Jordan canonical form J easily.

Example 1

Let V be the vector space of polynomial functions in two real variables x and y of degree at most 2, and T is the linear operator on V defined by

$$T(f(x,y)) = \frac{\partial}{\partial x} f(x,y) + \frac{\partial}{\partial y} f(x,y) \quad \forall f(x,y) \in V.$$

Find a Jordan canonical form J of T and a Jordan canonical basis β for T.

Solution

Let $\alpha = \{1, x, y, x^2, y^2, xy\}$. Then we denote $A = [T]_\alpha = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Since A is upper triangular, so the characteristic polynomial of A is $f(t) = t^6$. Then, there is only one eigenvalue

$$\lambda = 0 \quad \text{with} \quad m = 6.$$

Note, $\dim(E_\lambda) = 6 - \text{rank}(A - \lambda I) = 3$. Therefore, the number of columns in the dot diagram is 3. Moreover, as $(A-\lambda I)^2 = \begin{pmatrix} 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, so $\text{nullity}((A-\lambda I)^2) = 5$. Therefore, the resulting dot diagram for $\lambda = 0$ is

$$\begin{array}{lll} \bullet (A-\lambda I)^2 v_1 & \bullet (A-\lambda I) v_2 & \bullet v_3 \\ \bullet (A-\lambda I) v_1 & \bullet v_2 & \\ \bullet v_1 & & \end{array}$$

As the basis of K_λ consists of the vectors listed in the dot diagram, so we need to find end vectors v_1, v_2, v_3 such that the six vectors in the above dot diagram is non-zero and linearly independent.

Note,

$$N(A-\lambda I) = N\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\},$$

$$N((A-\lambda I)^2) = N\begin{pmatrix} 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

and

$$N((A-\lambda I)^3) = N(0) = \text{span}\{e_1, e_2, e_3, e_4, e_5, e_6\}.$$

By the definition of cycles, we have to choose $v_1 \in N((T-\lambda I)^3)$ but $v_1 \notin N((T-\lambda I)^2)$ and $v_1 \notin N(T-\lambda I)$. Similarly, $v_2 \in N((T-\lambda I)^2)$ but $v_2 \notin N(T-\lambda I)$, and $v_3 \in N(T-\lambda I)$. Therefore, we can choose

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{then} \quad (A-\lambda I)v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, (A-\lambda I)^2 v_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 2 \end{pmatrix}, \quad \text{then} \quad (A-\lambda I)v_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Finally, we can conclude that $J = Q^{-1}AQ$, where

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix},$$

and $\beta = \{2, x+y, xy, -x+y, -x^2+xy, -x^2-y^2+2xy\}$.

Example 2

Let T be a linear operator on a vector space V, and let $\gamma = \{(T-\lambda I)^{p-1}(x), (T-\lambda I)^{p-2}(x), \ldots, x\}$ be a cycle of generalized eigenvectors that corresponds to the eigenvalue λ. Prove that span(γ) is a T-invariant subspace of V.

Solution

Denote $W = \text{span}(\gamma)$. We need to show that W is a subspace of V, and W is T-invariant. To show W is a subspace of V is simple. Note,

1. $0 \in W$.
2. (closed under addition) For any $v, w \in W$, we can write $v = \sum_{i=1}^{p} a_i (T-\lambda I)^{p-i}(x)$ and $w = \sum_{i=1}^{p} b_i (T-\lambda I)^{p-i}(x)$. Then,

$$v + w = \sum_{i=1}^{p} a_i (T-\lambda I)^{p-i}(x) + \sum_{i=1}^{p} b_i (T-\lambda I)^{p-i}(x) = \sum_{i=1}^{p} (a_i + b_i)(T-\lambda I)^{p-i}(x) \in W.$$

3. (closed under scalar multiplication) For any $v \in W$, for any $c \in \mathbb{F}$,

$$cv = \sum_{i=1}^{p} c a_i (T-\lambda I)^{p-i}(x) \in W.$$

Before we show W is T-invariant, we first observe that W is $(T-\lambda I)$-invariant by the definition of a cycle. So, for any $v \in W$,

$$T(v) = (T-\lambda I)(v) + \lambda I(v) = (T-\lambda I)(v) + \lambda v \in W.$$

2 Exercises:

Question 1 (Section 7.2 Q5(d)):

Let T be a linear operator on $M_{2\times 2}(\mathbb{R})$ defined by $T(A) = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} A - A^T$ for any $A \in M_{2\times 2}(\mathbb{R})$. Find a Jordan canonical form J of T and a Jordan canonical basis β for T.

Question 2 (Section 7.2 Q7):

Let A be an $n \times n$ matrix whose characteristic polynomial splits and has k eigenvalues, γ be a cycle of generalized eigenvectors corresponding to an eigenvalue λ, and W be the subspace spanned by γ. Define γ' to be the ordered set obtained from γ by reversing the order of the vectors in γ.

(a) Let $T = L_A$. Prove that $[T_W]_{\gamma'} = ([T_W]_\gamma)^T$.

(b) Let J be the Jordan canonical form of A. Use (a) to prove that J and J^T are similar.

(c) Use (b) to prove that A and A^T are similar.

Solution

(For Question 1, please refer to Practice Problem Set 11.)

Question 2

(a) We denote $\gamma = \{v_1, v_2, \ldots, v_p\}$ where $v_i = (A-\lambda I)^{p-i}(x)$ and x is the end vector. Then, $\gamma' = \{w_1, w_2, \ldots, w_p\}$ where $w_i = v_{p-i+1}$. Note,

$$\begin{aligned}(T-\lambda I)(v_1) = 0 &\Rightarrow T(v_1) = \lambda v_1 \\ (T-\lambda I)(v_2) = v_1 &\Rightarrow T(v_2) = v_1 + \lambda v_2 \\ &\vdots \\ (T-\lambda I)(v_p) = v_{p-1} &\Rightarrow T(v_p) = v_{p-1} + \lambda v_p\end{aligned}$$

So, $[T_W]_\gamma = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}$. Similarly,

$$\begin{aligned} (T-\lambda I)(w_1) &= w_2 \Rightarrow T(w_1) = w_2 + \lambda w_1 \\ (T-\lambda I)(w_2) &= w_3 \Rightarrow T(w_2) = w_3 + \lambda w_2 \\ &\vdots \\ (T-\lambda I)(w_p) &= 0 \Rightarrow T(w_p) = \lambda w_p \end{aligned}$$

Therefore, $[T_W]_{\gamma'} = \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & 1 & \lambda & & \\ & & & \ddots & \\ & & & 1 & \lambda \end{pmatrix} = ([T_W]_\gamma)^T$.

(b) Since J is the Jordan canonical form of A, so there exists an invertible matrix $Q = (q_1, q_2, \ldots, q_n)$ such that $J = Q^{-1}AQ$. Form the result of (a), $J^T = P^{-1}AP$ where $P = (\gamma_1', \gamma_2', \ldots, \gamma_k')$ and γ_i is the set of generalized eigenvectors of λ_i. Therefore, J^T and J are similar.

(c) Since J is the Jordan canonical form of A, so J and A are similar. By (b), we have J and J^T are similar. Finally, J and A are similar implies J^T and A^T are similar. Therefore, A^T and A are similar.